Selected topics in geometry

- Euler Character formula
  App. only five regular polyhedrons
- Curvature of curves
  App. four vertex theorem
- Gauss-Bonnet (intrinsic surface theory)
  $$\int_{\Sigma} K\, dA = 2\pi\chi(\Sigma)$$
- Hopf's theorem (extrinsic surface theory)
  S^2 type embedded or immersed surface w/ constant mean curvature in R^3 must be the standard sphere
- Riemann's introduction of his curvature

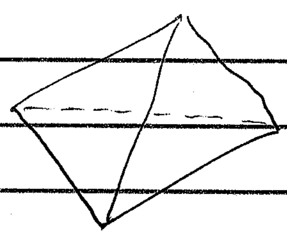
Euler Characteristic formula

An algebraic relation between the numbers of faces, edges, and vertices of a convex polyhedron.

Def. A convex polyhedron is the boundary of a convex bod in R^3 such that

- each piece of the boundary is a plane convex polygon (two-cell)
- those two-cells only meet along their sole common edge or vertex.

eg. A tetrahedron (pyramid)



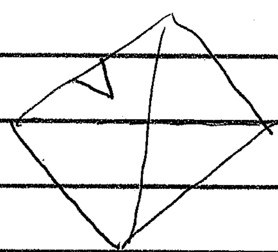
dim 0	dim 1	dim 2
$V=4$	$e=6$	$f=4$

$4+6+4$ No

$4-6-4$ No

$4-6+4=2$ Yes

c-eg.

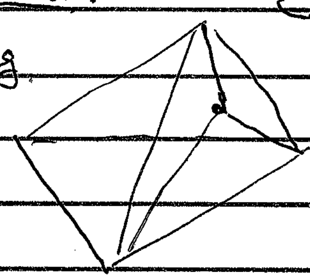


Not a polyhedron

A reasonable ~~reflection~~ should be an algebraic sum, and definite not an absolute sum of the numbers of v, e, f (otherwise, the sum could be arbitrarily large as v, e, f increase).

Euler. For any convex polyhedron, one has $v-e+f=2$!

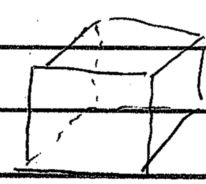
eg.



before $4-6+4=2$

Now $5-9+6=2$

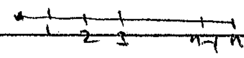
eg



Cube

$8-12+6=$

To motivate the proof, in fact as some preparation of Steiner's proof, we start from 1d & 2d situations.

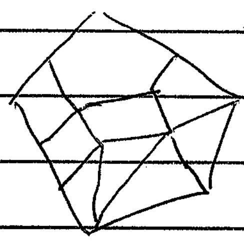
1-d. any n-cut of a segament 

$$v = n+1, \quad e = n, \quad v - e = 1$$

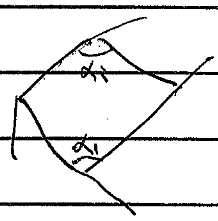
interior $v' = n-1$, interior $e' = n$, $v' - e' = -1$.

2-d. Any plane convex polygon (2-cell) cut into 2-cells C_i.
Let v, e, f denote the number of vertices, edges, and faces;
and v', e', f' ———————— interior ones;
N denote the number of possibly new vertices & edges of the boundary of the original 2-cell.

eg.



For each 2-cell C_i w/ n_i sides & angles α_i, one has



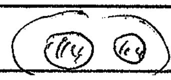
$$\sum (\pi - \alpha_i) = 2\pi \iff \sum_i \alpha_i = n_i \pi - 2\pi .$$

Summing over all 2-cells $\quad (N-2)\pi + V'2\pi = \underbrace{N\pi}_{\text{bdry edge}} + \underbrace{2e'\pi}_{\text{interior edge}} - 2\pi f'$

or $\boxed{v' - e' + f' = 1}$

note $v = v' + N$, $e = e' + N$, & $f = f'$

we have $\boxed{v - e + f = 1}$.

Rmk. Going further, we have the Euler characteristic of plane domains bounded or unbounded, closed or open w/ n-holes being $\chi = 1 - n$.  

Now we present Legendre's proof of Euler characteristic formula.

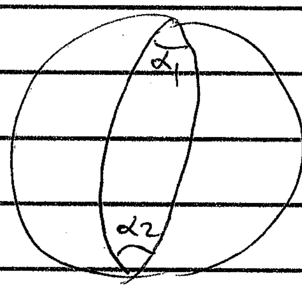
step1. project to the unit sphere

Project the convex polyhedron from an interior point to the unit sphere around that interior point (convexity is used). Then we obtain a spherical polyhedron on the unit sphere w/ each edge being a GREAT CIRCLE! while the numbers of vertices, edges, and faces remain the same by convexity.

step2. spherical triangle / n-polygon interior angle identity

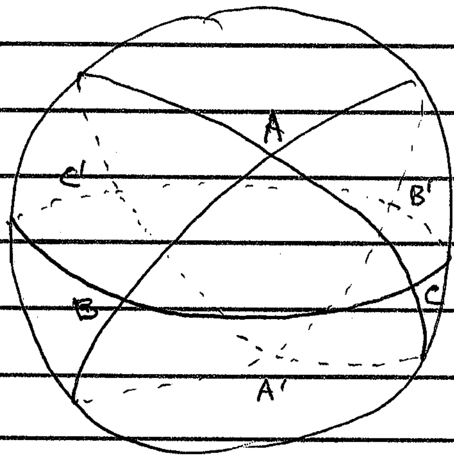
$$\sum_i \alpha_i = (n-2)\pi + \text{area}$$

$n=2$ The two interior angles are the same, as the two-poly (called lune) is symmetric w.r.t. the plane through the center & perpendicular to the diameter between the two vertices of this 2-cell



$\text{area} = \frac{\alpha_1}{2\pi}\cdot 4\pi = \alpha_1 + \alpha_2$

$n=3$ (Girard's proof)



Lunes formed by plane / great circle

circle	AB & AC	L_A
	BA & BC	L_B
	CA & CB	L_C

Obs. $\triangle A'B'C'$ is the antipodal image of $\triangle ABC$

(point by point, then all great circles from A to BC)

obs. Covering of the sphere by three pair of antipodal lunes has the antipodal triangle pair $\triangle ABC$ & $\triangle A'B'C'$ appearing two extra times.

$$2L_A + 2L_B + 2L_C = 4\pi + 2\triangle ABC + 2\triangle A'B'C' = 4\pi + 4\triangle ABC$$

From $n=2$. $L_A = 2A$, $L_B = 2B$, & $L_C = 2C$. Then

$$A + B + C = \pi + \text{area}$$

or

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + \text{area}$$

RMK. We will reprove this formula using calculus on general surfaces

$n > 3$ Induction. We divide ~~the~~ spherical n-polygon into a triangle and an $(n-1)$-polygon; rather subdivide the corresponding convex flat n-polygon, then project to the unit sphere (this way, one sees the dividing great circle is inside the spherical n-polygon)

$$\sum \alpha_i = (n-1-2)\pi + \text{area of } (n-1)\text{-polygon} + \underbrace{\alpha'_{n-2} + \alpha_{n-1} + \alpha'_n}_{\pi + \text{area of triangle}} + \cancel{\text{area of 3-polygon}}$$

$$= (n-2)\pi + \text{area}.$$

Step 3. Summing in two different ways

Similar to the 2-d case, but now w/ the extra area term added for each spherical 2-cell C_i w/ n_i sides & area A_i

$$\sum_i \alpha_i = (n_i - 2)\pi + A_i$$

Sum over all 2-cells $\sum_{i,j} \alpha_{i,j} = \sum_j (n_{i,j} - 2)\pi + \sum_j A_{i,j}$

$$v\,2\pi = 2e\pi - f\,2\pi + 4\pi$$

OR $v - e + f = 2$

Application. There are only 5 regular polyhedrons (4, 6, 8, 12, & 2[illegible] faces). A regular polyhedron means all its faces have the same $m^{\geq 3}$ amount of edeges & vertices, all its vert[illegible] have the same n amount of faces & edges, and also all faces are congruent.

The analysis is to substitute f & v in terms of m, n, e in Euler formula, then solve for m, n, e. Going w/ vertices v or faces f is complicated, one reason is the loss of symmetry of m & n in the equation.

Obs. vertex $v = 2e/n$

edge $e = mf/2$ or $f = 2e/m$

Euler $\frac{2e}{n} - e + \frac{2e}{m} = 2$

or $\frac{1}{m} + \frac{1}{n} = \frac{1}{2} + \frac{1}{e}$

Now $\frac{1}{m} + \frac{1}{n} > \frac{1}{2} \overset{m,n\geq 3}{\Longrightarrow}$ at least one of m & n is 3.

$n = 3$ $\frac{1}{m} = \frac{1}{6} + \frac{1}{e}$ $3 \leq m < 6$

$m = 3$ $e = 6$, $f = 4 = v$

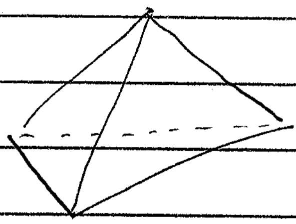
$m = 4$ $e = 12$, $f = 6$, $v = 8$

$m = 5$ $e = 30$, $f = 12$, $v = 20$

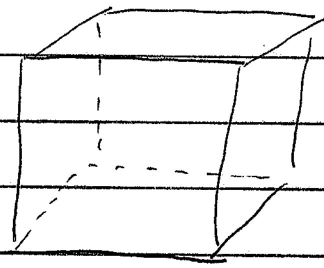
$m = 3$ is symmetric

$m \backslash n$	3	4	5
3	(4, 6, 4) (v, e, f)	(6, 12, 8)	(12, 30, 20)
4	(8, 12, 6)		
5	(20, 30, 12)		

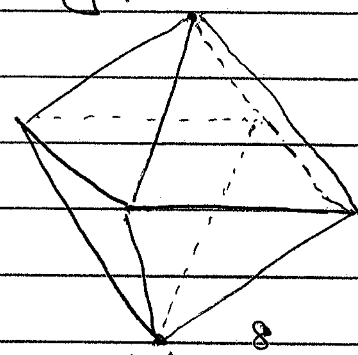
Finally using the congruent assumption of faces, we see that 5 regular polyhedrons are unique upto scaling.



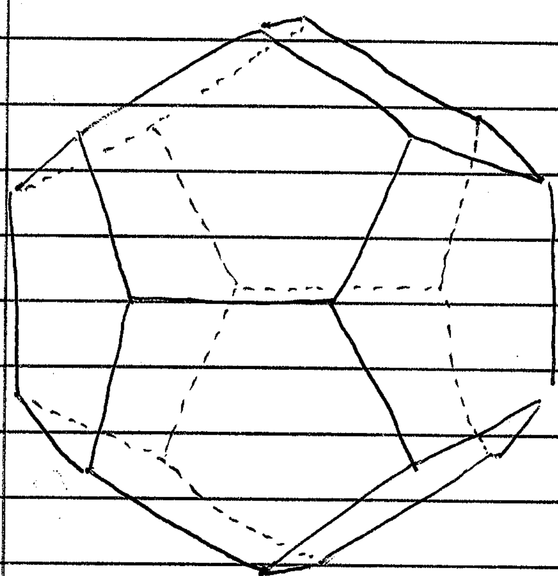
4
Tetrahedron



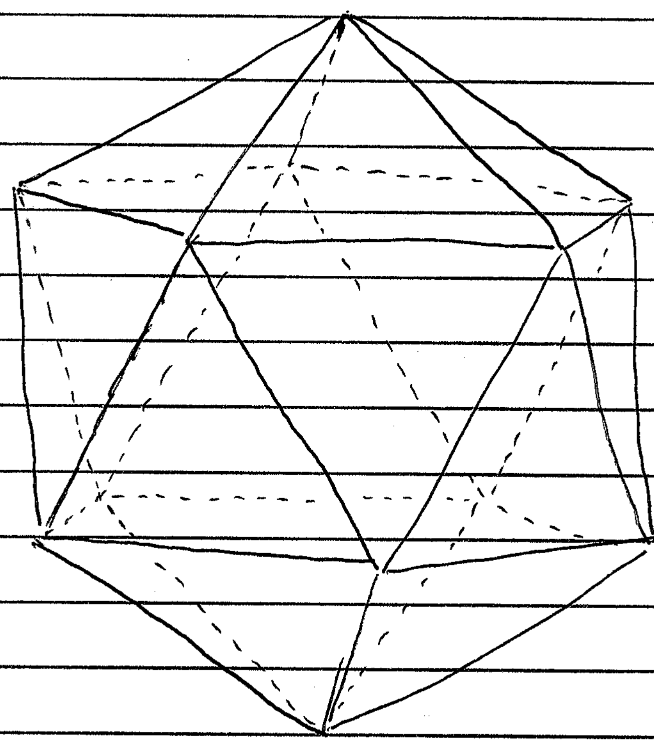
6
Hexahedron



8
Octahedron



12
Dodecahedron



20
Icosahedron

Steiner's proof of Euler Characteristic formula

The argument is to show the Combination of two Convex polyhedrons on the unit sphere is a subdivision of both corresponding spherical polyhedrons. The subdivision doesn't change the Euler characteristic. Finally take one simplest polyhedron say a tetrahedron, which results the common Euler characteristic $4-6+4=2$.

Step1. Recall a polyhedron (spherical or not) satisfies: any pair of its two-cells only meet along their sole common edge or vert (otherwise, they don't intersect)

A subdivision of a polyhedron is still a polyhedron.

Thm. Let polyhedron P_2 be a subdivision of polyhedron P_1

Then $\chi(P_2)=\chi(P_1)$.

Proof. • χ no change on edges

On any edge of P_1 where new interior vertices & edges are introdu the contribution of new vertices & edges is

$$v'-e'=-1$$

This edge contributes -1 to $\chi(P_1)$ originally too.

• χ no chang on cells

On any two cell of P_1 where new interior vertices, edges, and 2-cells are introduced, recall from our first 2-d consideratio the new interior v', e', f' contribute to the new $\chi(P_2)$ is

$$v'-e'+f'=1.$$

This 2-cell contributes 1 to $\chi(P_1)$ originally too.

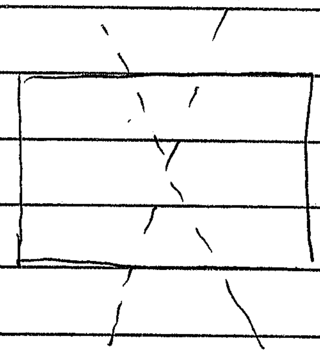
Therefore the change of $\chi(P_1)$ to $\chi(P_2)$ is none.

Step 2. Combination

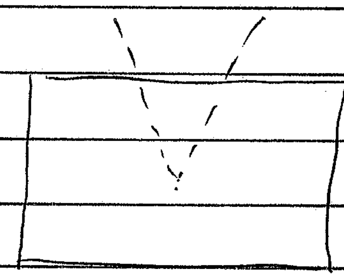
project both given convex polyhedron P and a tetrahedron T on the unit sphere. Then the "Combination" of projections C is a subdivision of both projections. From step 1

$$\chi(P)=\chi(C)=\chi(T)=4-6+4=2.$$

eg



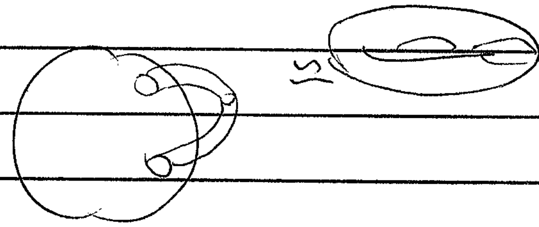
c-eg

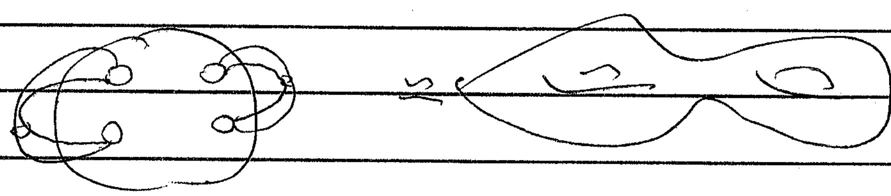


Surfaces of higher genus

A closed (bounded, no boundary) surface of genus p can be obtained by cutting out 2p (small) disks from a sphere and connecting the boundary circles pairwise so that they (new handles) don't intersect.

eg.





It can be proved that all topologically equivalent (namely homeomorphic) closed orientable surfaces have the same genus and the single number of genus determines homeomorphical equivalent classes of all closed orientable surfaces.

Thm. All networks (curved polygons) on a surface of genus p have the same characteristic $\chi(p)$

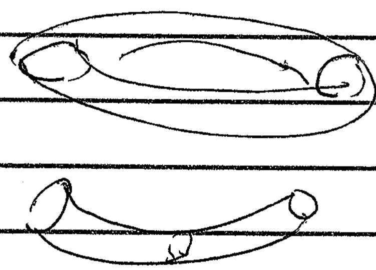
Idea of proof. Similar to the genus 0 case, given networks P & Q on a genus p close surface, we "combine" P & Q to form a subdivision S of both P & Q. Hence

$$\chi(P) = \chi(S) = \chi(Q) \overset{\text{define}}{=\!=} \chi(\text{genus } p \text{ close surface})$$

Finally we inductively calculate $\chi(p) = 2 - 2p$.

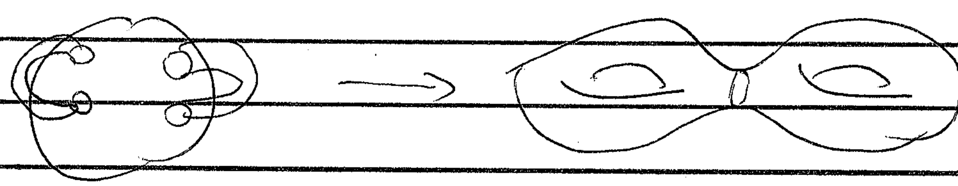
$p=0$ S^2 $\chi(0) = \chi(S^2) = 2$

$p=1$ T^2



Cut the torus into two parts, each part is equivalent to a sphere w/ two 2-cells removed, thus each part will have characteristic 0. Putting back the two parts will not alter the ~~same~~ characteristic since there are the same number of edges and vertices along the cut which contribute nothing to χ. Thus $\chi(1) = \chi(\text{Torus}) = 0$

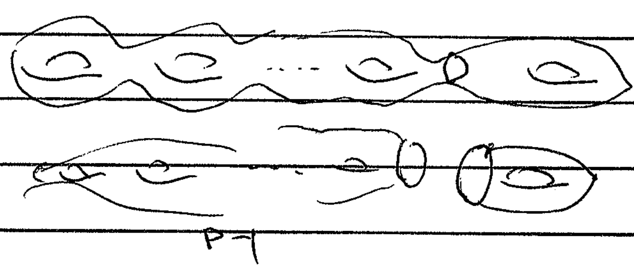
$p=2$



First move (topologically) the next handle on top of the first one, we can assume the genus 2 surface is a torus w/ two holes. Now cut the two hole torus along the middle neck into 2 parts, each part is genus 1 torus w/ one two cell removed, that has $\chi = -1$. Adding back two parts will not change the characteristic, again as there are the same number of edges & vertices along the cut which contribute nothing to χ

Thus $\chi(2) = \chi(\text{torus w/ two handles}) = 0 - 1 - 1 = -2$

$p > 2$ similary (move handles over handles, cutting along the (last) neck, we decompose genus p surface into a $(p-1)$-hole torus and a 1-hole torus, each w/ one 2-cell removed



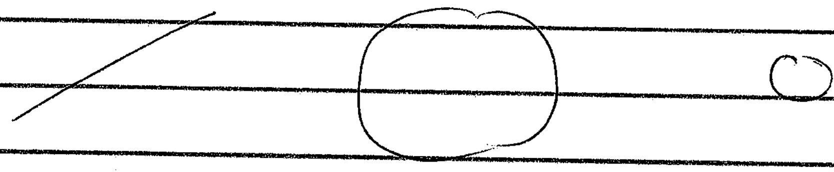
$\chi = \chi(p-1) - 1$ $\qquad$ $\chi = 0 - 1$

Thus $\chi(p) = \chi(p-1) - 2 = 2 - 2p$

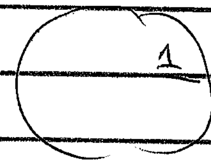
Curvature of Curves

A second derivative property; describing how much a wavey curve curves, or how fast its tangent direction changes.

eg.

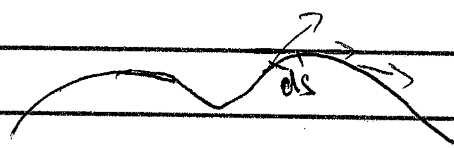


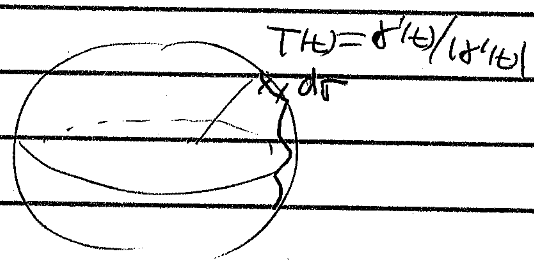
eg. $(\cos t, \sin t)$ $(\cos 2t, \sin 2t)$



Geometric definition.

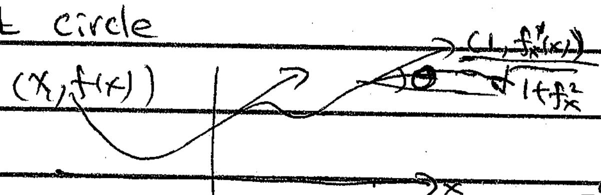
The (absolute) curvature of a C^2 curve $\gamma(t) \in \mathbb{R}^3$ at t_0 is the infinitesimal ratio between the arclength of ~~the~~ spherical image of the unit tangent of the curve and the arclength of the curve at t_0.

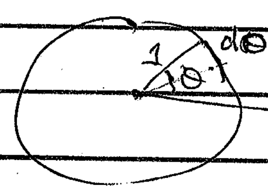




$$\kappa = \frac{d\sigma}{ds}$$

In particular for plane curves, the spherical image falls on a unit circle





$$\kappa = \frac{d\theta}{ds} = \frac{d}{ds}\arctan f_x = \frac{\frac{d}{dx}\arctan f_x}{\frac{ds}{dx}}$$

$$= \frac{f_{xx}}{\left(\sqrt{1+f_x^2}\right)^3} = \left(\frac{f_x}{\sqrt{1+f_x^2}}\right)_x$$

Analytic expression

$$\kappa(t) = \frac{|\gamma_{tt} \times \gamma_t|}{|\gamma_t|^3}$$

- $\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{\frac{d}{dt}\frac{\gamma'(t)}{|\gamma'(t)|}}{\frac{ds}{dt}}\right| = \frac{\left|\frac{\gamma''(t)}{|\gamma'(t)|} + \gamma'(t)\left(\frac{1}{|\gamma'(t)|}\right)'\right|}{|\gamma'(t)|}$

- $T_s \perp T$, then $|T_s| = |T_s||T| \sin\frac{\pi}{2} = |T_s \times T|$
- $\kappa = \left|\left[\frac{\gamma''(t)}{|\gamma'(t)|} + \gamma'(t)\left(\frac{1}{|\gamma'(t)|}\right)'\right] \times \frac{\gamma'(t)}{|\gamma'(t)|}\right| / |\gamma'(t)| = \frac{|\gamma'' \times \gamma'|}{|\gamma'(t)|^3}$.

Other interpretation of curvature

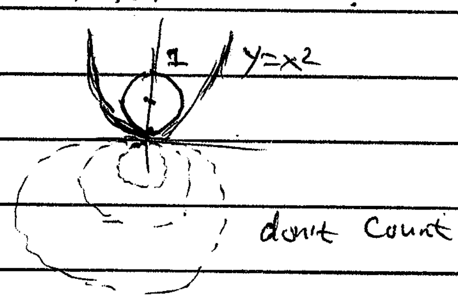
physically, when speed unit, magnitude of acceleration is the curvature of the orbit/trace/curve.

Circle way, second order touching/kissing/osculating circle's radius is the reciprocal of the curvature.

Quadratic approximation: Say the curve is already a graph over its tangent line (can always achieve it via translat & rotation) $\gamma(x) = (x, g(x), h(x))$. Near the tangent point, the quadratic approximations of $g(x)$ & $h(x)$ take form $g(x) = \cancel{g(0)}^0 + \cancel{g'(0)}^0 x + \frac{1}{2}g''(0)x^2 + \cdots$, similar to h.

Then $\kappa(0) = \sqrt{[g''(0)]^2 + [h''(0)]^2}$

eg $(x, x^2, 0)$ $\kappa(0) = 2$.

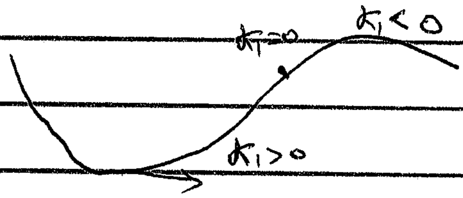


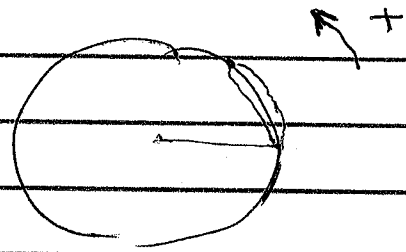
Total Curvature

The total curvature (or integral) between two points s_1 & s_2 arclength parametrization is the length of the spherical image curve

$$K = \int_{s_1}^{s_2} \kappa(s)ds = \int_{s_1}^{s_2} d\sigma = \text{length}.$$

For a plane curve, the spherical image now is on a great circle image. We can give an orientation to this great circle, then $d\sigma$ has a sign. We define signed curvature of a plane curve $\kappa_1 = \frac{d\sigma}{ds}$

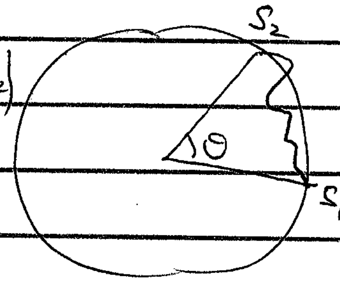




certainly $|\kappa_1| = \kappa$.

Come back to space curve, let $\theta(s_1, s_2) \in [0, \pi]$ denote the angle between $T(s_1)$ & $T(s_2)$. This angle is the spherical distance of the spherical image $T(s_1)$ & $T(s_2)$, namely the arc length of the great circle through $T(s_1)$, $T(s_2)$ & O. By the geodesic/minimality property of great circle, we have

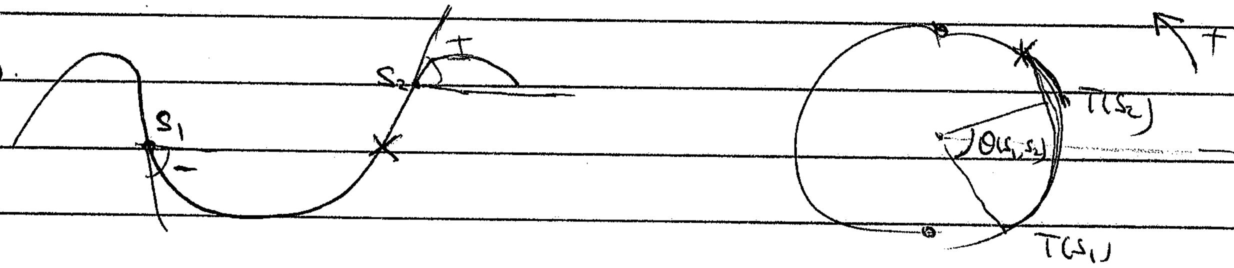
$$\theta(s_1, s_2) \leq \text{length of spherical curve } \{T(s) : s_1 \leq s \leq s_2\}$$
$$= \int_{s_1}^{s_2} \kappa(s)\, ds$$



The equality holds only for plane curves w/ a monotonically turning tangent of total curvature $\leq \pi$
(convex or concave curve)

For plane curve, a sufficient condition for the signed curvature satisfying
$$\theta(s_1, s_2) = \int_{s_1}^{s_2} \kappa_1(s)\, ds$$
(then inside half circle)
is the total curvature $\int_{s_1}^{s_2} \kappa(s)\, ds \leq \pi$ & the orientation of the great circle is compatible w/ the angle orientation

eg.

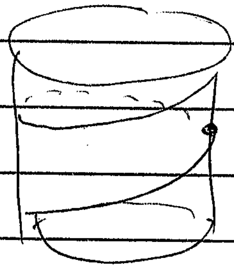


Lemma. Two plane curves w/ the same signed (scalar) curvature for arclength parameter and starting w/ the same position & direction are identical.

Rmk. Same initial position & velocity vector, and same acceleration vector at all time determine a unique orbit.

c-eg. For space curves, same curvature doesn't determine the same curve (still w/ same initial position & tangent direction) or doesn't imply congruence of curves.

Helix



$\gamma(t) = (\cos t, \sin t, t)$

$\gamma' = (-\sin t, \cos t, 1)$

$\gamma'' = (-\cos t, -\sin t, 0)$

$\kappa = \frac{|\gamma'' \times \gamma'|}{|\gamma'|^3} = \frac{1 \cdot \sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}$

~~$\delta(t)$ circle w/ radius 2~~

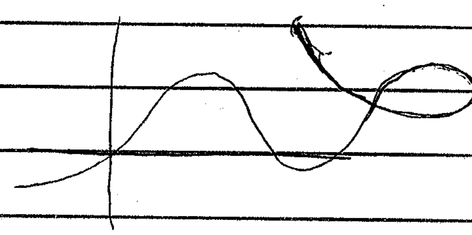
proof of Lemma.

We give a construction of the curve, at the same time, the uniqueness follows.

As $\gamma_s(s) = (x_s(s), y_s(s))$ has unit length, we may assume $x_s(0)$; even $x_s(0) > 0$.

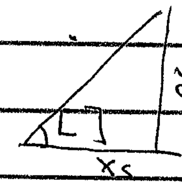
Recall

$\frac{d\theta}{ds} = \frac{d}{ds} \arctan \frac{y_s}{x_s} = \kappa_1(s)$



then $\arctan\frac{y_s}{x_s} = \int_0^s k_1(\tau)d\tau + \theta(0)$ $\quad$ // $\arctan\frac{y_s(0)}{x_s(0)}$

Hence $\frac{y_s}{x_s} = \tan\left[\int \quad + \theta(0)\right]$.



Note $y_s = \pm\sqrt{1-x_s^2}$, the sign is determined by the sign of $y_s(0)$, say $+$.

Then $x_s(s) = \cos\left[\int_0^s k_1(\tau)d\tau + \theta(0)\right]$

$y_s(s) = \sin\left[\int_0^s k_1(\tau)d\tau + \theta(0)\right]$

and

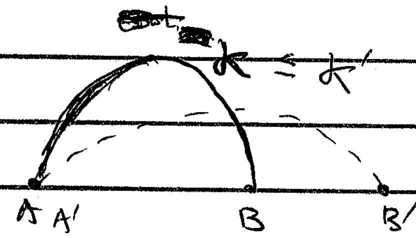
$$\begin{cases} x(s) = \int_0^s \cos[\quad]\,dt + x(0) \\ y(s) = \int_0^s \sin[\quad]\,dt + y(0) \end{cases}$$

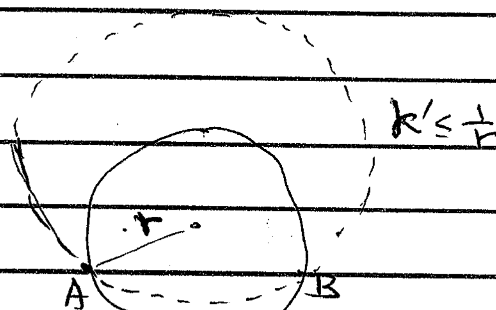
When x_s becomes 0, we "restart" / work w/ y_s first. This way we completely & uniquely determine the plane curve via its signed curvature.

Comparison of chord or length under curvature size relation.

Schur: same arclength, the less curvature, the larger chord

Schwartz: Same chord, the less curvature, the more "spread" of arclength.





Either $L' \geq Arc_L$ or $L' \leq Arc_S$

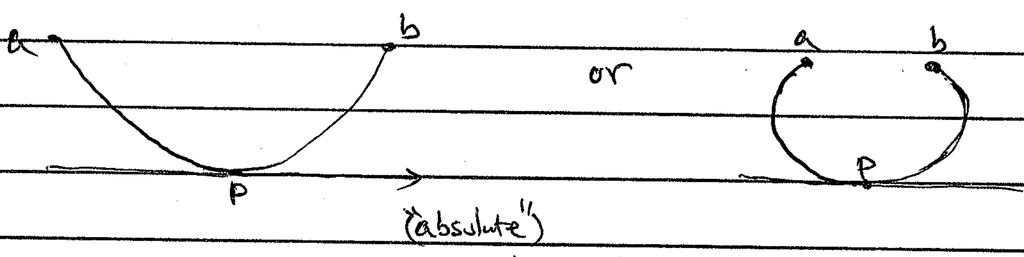
But no $Arc_S < L' < Arc_L$

Thm (Schur) Given a model plane convex C^2 curve C w/ end points a & b. For any space or plane curve C' w/ end points a' & b' and the same arclength ℓ as C. Let $\kappa(s)$ & $\kappa'(s)$ be the curvatures of C & C' w/ arclength parameter s starting from a & a' respectively. Suppose

$$\kappa'(s) \le \kappa(s).$$

THEN the chord of C' $\overline{a'b'} \ge$ the chord of C $\overline{ab}$; and the equality holds if and only if C & C' are congruent.

proof. On the model convex curve C, there exists a (unique) point p w/ arclength from a being s_1 such that the tangent at $C(s_1)$ is parallel to chord ab.



Recall $\theta(s_1, s)$ denotes the angle between tangents $T(s)$ & $T(s_1)$, and

$$\theta(s_1,s) = \left|\int_s^{s_1} \kappa(z)\,dz\right| \quad \text{really} \quad \begin{cases} +\int_s^{s_1} & s \le s_1 \\ -\int_s^{s_1} & s > s_1 \end{cases}$$

Note $0 \le \theta(s_1, s) \le \pi$ (But $\theta(0, \ell)$ may $> \pi$)

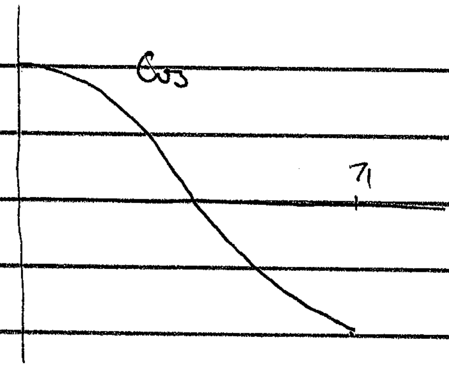
Further for space curve C'

$$\theta'(s_1,s) \le \left|\int_s^{s_1} \kappa'(z)\,dz\right|$$

geodesic $\le$ tangent image length on sphere

By the curvature assumption $\kappa' \le \kappa$, it follows ~~that~~

$$0 \le \theta'(s_1,s) \le \theta(s_1,s) \le \pi.$$



Take Cosine of both sides (remember Cos ↓ on $[0,\pi]$)

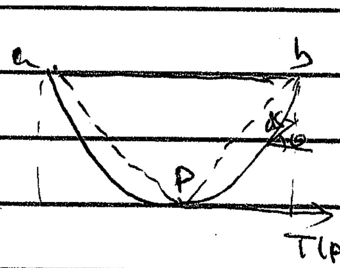
$$\cos\theta'(s_1,s) \geq \cos\theta(s_1,s)$$

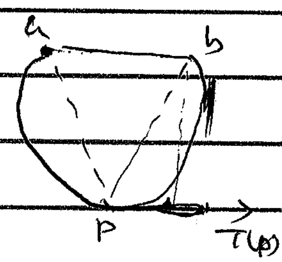
Integrate both sides from 0 to ℓ

$$\int_0^\ell \cos\theta'(s_1,s)\,ds \geq \int_0^\ell \cos\theta(s_1,s)\,ds .$$

Model Right hand Side first. The RHS is the length of projection of chord ab to the tangent $T(s_1)$ at P. As $T(s_1)$ is parrelle to ab, then

$$\text{RHS} = \overline{ab}$$



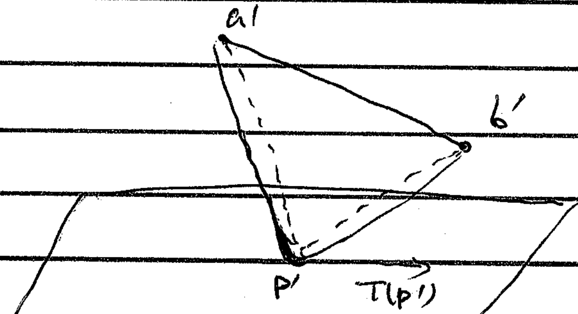


signed projection on tangent P

Also $\text{RHS} = \operatorname{Comp}_{T(p)} \overrightarrow{Pb} + \operatorname{Comp}_{T(p)} \overrightarrow{ap} = \operatorname{Comp}_{T(p)} \overrightarrow{ab}$

space curve Left hand Side.

Claim. LHS = the length of projection of chord a'b' to the tangent $T'(s_1)$ at p'.



observe any perpendicular direction to $T(p')=T(s_1)$ is non-effecti
direction (for inner product $\langle T'(s), T(s_1)\rangle = \cos\theta'(s_1,s)$) to the project
on to $T(p')$, in particular take away the one $\perp T(p') \,\&\, p'b'$
and also the one to $T(p') \,\&\, p'a'$. To put it in another way, w
may project $p'b'$ part of the space curve to $T(p') \,\&\, p'b'$ plane
w/o changing the integral of $\int_{s_1}^{\ell} \cos\theta'(s_1,s)\,ds$. Then argue
as in the above plane case, we see the $\int_{s_1}^{\ell}$ part of LHS
is $\mathrm{Comp}_{T(p')}\, \overrightarrow{p'b'}$. Similarly the $\int_0^{s_1}$ part of RHS is $\mathrm{Comp}_{T(p')}\, \overrightarrow{a'p'}$

As $\overrightarrow{a'p'}+\overrightarrow{p'b'}=\overrightarrow{a'b'}$, the claim holds.

Therefore we have proved

$$\text{chord } \overline{a'b'} \quad \left(\geq \mathrm{Comp}_{T(p')}\, \overline{a'b'}\right) \geq \text{chord } \overline{ab}.$$

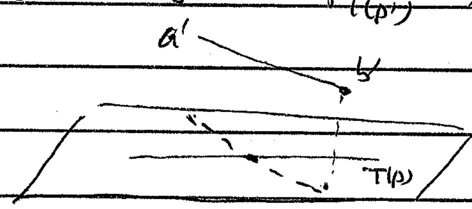
The equality $\overline{a'b'}=\overline{ab}$ holds if and only if ~~c could still be~~ two different p

a) the arc $\widehat{a'p'}$ & $\widehat{p'b'}$ must be plane curves, as the equality holds for the angle-curvature inequality for space curve

b) $\kappa'(s) \equiv \kappa(s)$

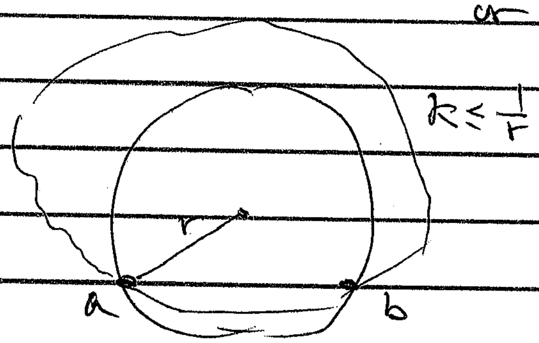
c) $\overline{a'b'} = \mathrm{Comp}_{T(p')}\, \overline{a'b'}$

From the curvature-determine-plane curve Lemma, arc $\widehat{a'p'}$ & $\widehat{p'b'}$ are congruent to the plane arc $\widehat{ap}$ & $\widehat{pb}$ respectively, and those two parts are in plane $a'p'-T(p')$ & $p'b'-T(p')$ resp.
Lastly we justify arc $\widehat{a'p'}$ & arc $\widehat{p'b'}$ are in the same plane. This is just because $\overline{a'b'} = \mathrm{Comp}_{T(p')}\, \overline{a'b'}$, one must have $a'b' \parallel T(p')$.
Then those two planes coincide.



∎

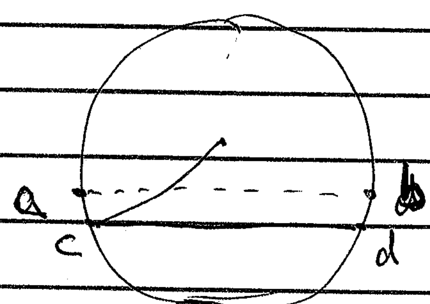
Thm (Schwartz) Given a plane or space curve of length L share the same chord w/ a circle of radius r. If the curvature of the curve $k(s) \le 1/r$. THEN either $L \ge$ the bigger arc of circle
or $L \le$ the smaller arc of circle



proof. Case $L \ge 2\pi r$, then we're done.

Case $L < 2\pi r$, we still prove either $L \ge$ bigger arc
or $L \le$ small arc.

On the circle of radius r, we cut an arc of length L w/ chord cd (doable since $L < 2\pi r$), either top or bottom of cd



Now we have Schur situation. same length of L for circle arc $\widehat{cd}$ or $\underset{\smile}{cd}$ & original curve $\widehat{ab}$, and $k \le \frac{1}{r}$.

Then chord $\overline{ab} \ge \overline{cd}$

Draw parallel chord ab to cd on the "new" circle, it is closer to the center. Therefor

$L = \widehat{cd} \ge$ bigger arc $\widehat{ab}$, if arc $\widehat{cd}$ on top

$L = \underset{\smile}{cd} \le$ smaller arc $\underset{\smile}{ab}$, if arc $\underset{\smile}{cd}$ on bottom.

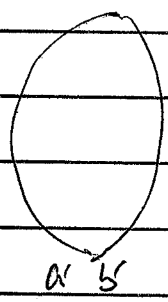
Shortest ring w/ bounded curvature at most κ_0

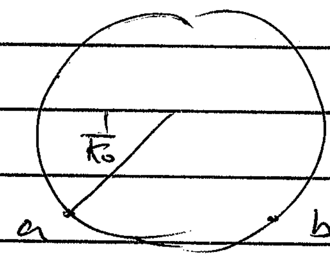
Consider a physical problem: finding the shortest piece of wir[e] the end points of which can be brought together w/o break[ing] the wire, namely without increasing its curvature at any point beyond κ_0.

Another formulation: build a shortest track for car running on it w/ constant unit speed and (center-seeking) acceleration at most κ_0.

Answer. The shortest one is the circle w/ radius $1/\kappa_0$.

Suppose there is a closed curve (space or plane) of arclength strictly less than $2\pi/\kappa_0$ (the two endpoints may not be connected smoothly, elsewhere is smooth) and curvature $\le \kappa_0$. Lets find some inconsistency.

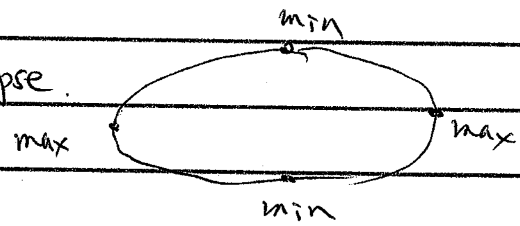




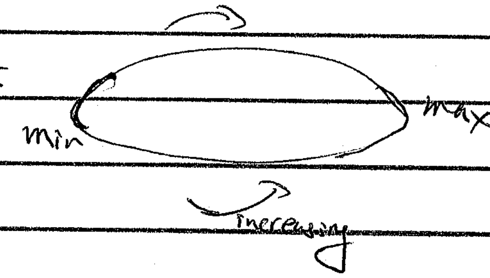
On the circle of radius $\frac{1}{\kappa_0}$, cut an arc $\widehat{ab}$ w/ length L (as $L < 2\pi\frac{1}{\kappa_0}$). Its chord $\overline{ab} > 0$. But according to Schure the two endpoints of the ~~shorter~~ same length L ~~arc~~ curve C' a' & b' should have larger distance (chord length $\overline{a'b'}$) than chord $\overline{ab}$. But $\overline{a'b'} = 0$. This is a contradiction.

Four-vertex-Thm. The curvature of a closed simple (no self intersection) convex (plane) curve has at least four extrem

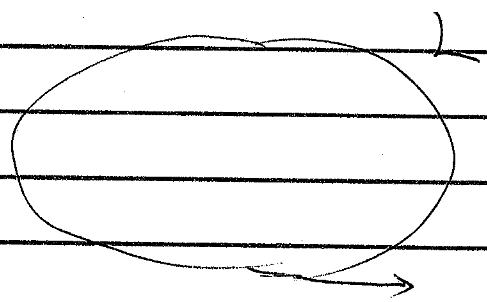
eg. ellipse

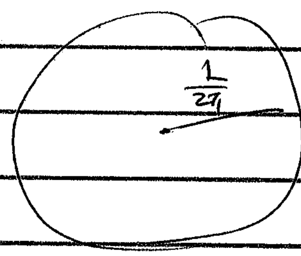


not



proof. The argument is to compare the curve to the model circle w/ the same length L in showing their curvatures coinside at least four times. Between those four points ~~there~~ are at least four local extremal values of curvature for the curve.





At least for simple convex close curves, by the monotonic property of the angle between tangents and a fixed one it is clear

$$\int_0^L \kappa(s)\,ds = 2\pi = \int_0^L \frac{2\pi}{L}\,ds$$

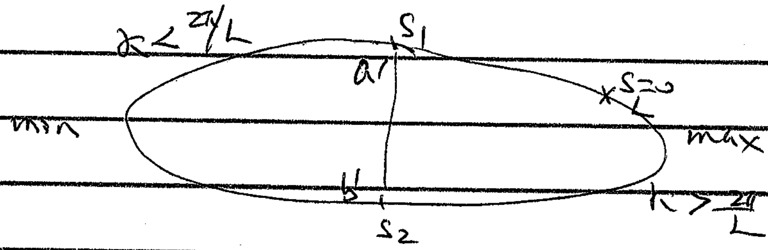
(True for all simple plane curves).

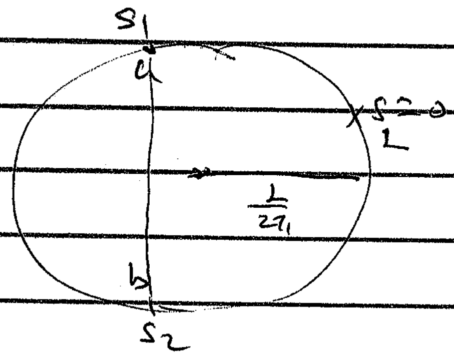
Rewrite

$$\int_0^L \left(\kappa(s) - \frac{2\pi}{L}\right) ds = 0$$

We observe that κ equals $2\pi/L$ at least once, in fact even number of times, as κ is a periodic function (anything goes up must come down). In the following we show the number cannot be two.

Suppose κ equals $\frac{L}{2\pi}$ only twice at s_1, s_2





Compare arc $\widehat{s_1 s_2}$ to circle portion $\widehat{s_1 s_2}$ $\xRightarrow{\text{Schur}}$ $\overline{a'b'} > \overline{ab}$

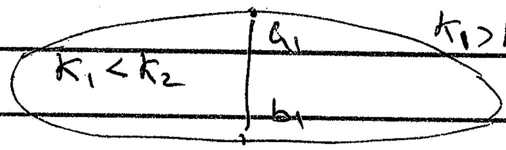
arc $\widehat{s_2 L s_1}$ to circle portion $\widehat{s_2 L s_1}$ $\xRightarrow{\text{Schur}}$ $\overline{ab} > \overline{a'b'}$.

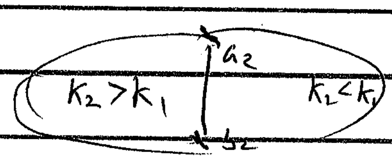
This contradiction shows κ coinside w/ $2\pi/L$ at four times. It follows ~~that~~ ~~there~~ are at least four local extreme value for κ. $\square$

In fact, exactly the same argument, allows us to replace ~~the~~ model circle w/ an arbitrary curve of the same length, the two curvatures of convex simple curves coinside at least four times. OR JUST compare each curve w/ CIRCLE of radius $\frac{L}{2\pi}$.

Thm. Given two simple convex curves w/ the same length. Then their curvatures coinside at least four times

proof. $\int_0^L [\kappa_1(s) - \kappa_2(s)] ds = 2\pi - 2\pi = 0$





$\square$